

# Self-bumping of deformation spaces of hyperbolic 3-manifolds

K. Bromberg\* and J. Holt†

September 4, 2000

## Abstract

Let  $N$  be a hyperbolic 3-manifold and  $B$  a component of the interior of  $AH(\pi_1(N))$ , the space of marked hyperbolic 3-manifolds homotopy equivalent to  $N$ . We will give topological conditions on  $N$  sufficient to give  $\rho \in \overline{B}$  such that for every small neighborhood  $V$  of  $\rho$ ,  $V \cap B$  is disconnected. This implies that  $\overline{B}$  is not manifold with boundary.

## 1 Introduction

In this paper we study aspects of the topology of deformation spaces of Kleinian groups. The basic object of study is  $AH(\pi_1(N))$ , the space of isometry classes of marked, complete hyperbolic 3-manifolds homotopy equivalent to  $N$ , where  $N$  is a compact, orientable, irreducible, atoroidal 3-manifold with boundary. The study of the global topology of  $AH(\pi_1(N))$  was begun by Anderson, Canary and McCullough in [2] for the case in which  $N$  has incompressible boundary. They described necessary and sufficient criteria for two components of the interior of  $AH(\pi_1(N))$  to “bump”; that is, to have intersecting closures. We address the question of when a component of the interior “self-bumps”; that is, if  $B$  denotes such a component, then when is there an element  $\rho$  in the closure of  $B$  such that for any sufficiently small neighborhood  $V$  of  $\rho$  in  $AH(\pi_1(N))$  the set  $V \cap B$  is disconnected? In this paper we will establish the following result:

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\*Partially supported by a grant from the Rackham School of Graduate Studies, University of Michigan and by the Clay Mathematics Institute

†Partially supported by a National Science Foundation Postdoctoral Fellowship

**Theorem 4.5** *Let  $N$  be a compact, orientable, atoroidal, irreducible 3-manifold with boundary. Suppose that  $N$  contains an essential, boundary incompressible annulus whose core curve is not homotopic into a torus boundary component of  $\partial N$ . Let  $B$  be a component of the interior of  $AH(\pi_1(N))$ . Then there is a representation  $\rho$  in  $\bar{B}$  such that for any sufficiently small neighborhood  $V$  of  $\rho$  in  $AH(\pi_1(N))$  the set  $V \cap B$  is disconnected.*

Note that this result applies even when  $N$  has compressible boundary.

In [11] McMullen, using projective structures and ideas of Anderson and Canary, proved Theorem 4.5 when  $N$  is an oriented  $I$ -bundle over a surface. Our techniques avoid the use of projective structures, and furthermore, even in the  $I$ -bundle case we will find bumping representations that are not detected with McMullen's methods. In a sequel, we will use the techniques developed here to study the topology of the space of projective structures with discrete holonomy.

We sketch the proof of Theorem 4.5 in the case where  $N = S \times [0, 1]$  is an  $I$ -bundle over a closed surface of genus  $\geq 2$ . In this case the interior of  $AH(\pi_1(N))$  consists of a single component of quasifuchsian structures on  $M = \text{int } N$ , which is usually denoted  $QF(S)$ .

To construct the representation where bumping occurs we start with a hyperbolic structure on  $M$  with a curve removed. That is choose a simple closed curve  $c$  on  $S$  and let  $\hat{M} = M - (c \times \{1/2\})$ . Then give  $\hat{M}$  a geometrically finite hyperbolic structure. Now,  $\pi_1(\hat{M})$  has many conjugacy classes of subgroups isomorphic to  $\pi_1(S)$ , for example  $S \times \{1/4\}$  and  $S \times \{3/4\}$  each define such a subgroup. However, to find our bumping representation we choose a non-standard subgroup of  $\pi_1(\hat{M})$  by wrapping  $S$  around the removed curve (see Figure 1). Then the hyperbolic structure on  $\hat{M}$  defines a representation of  $\pi_1(\hat{M})$  and our choice of subgroup defines a representation  $\rho_\infty$  of  $\pi_1(S)$ . The cover  $M_\infty$  associated to this subgroup will be homeomorphic to  $M$ .

The next step is to construct an immersion  $f : N \rightarrow \hat{M}_\infty$  in the homotopy class associated to  $\rho_\infty$  and then use  $f$  to pull back a hyperbolic structure  $N_\infty$  on  $N$ . For each  $\rho \in AH(\pi_1(N))$  there is a hyperbolic 3-manifold  $M_\rho$  homeomorphic to  $M$ . Given a small neighborhood  $V$  of  $\rho_\infty$ , for each  $\rho \in V$  a general theorem allows us to construct a smoothly varying family of hyperbolic structures  $N_\rho$  on the compact manifold  $N$ . Here  $N_\rho$  has holonomy  $\rho$  and  $N_{\rho_\infty} = N_\infty$ . Since  $N_\rho$  and  $M_\rho$  have the same holonomy there will be an isometric immersion  $f_\rho$  of  $N_\rho$  in  $M_\rho$ . If  $\rho \in V \cap QF(S)$  then  $c$  will have a geodesic representative  $c_\rho$  in  $M_\rho$  and there will be a canonical homeomorphism between  $M_\rho - c_\rho$  and  $\hat{M}$ . Furthermore, geometric consid-

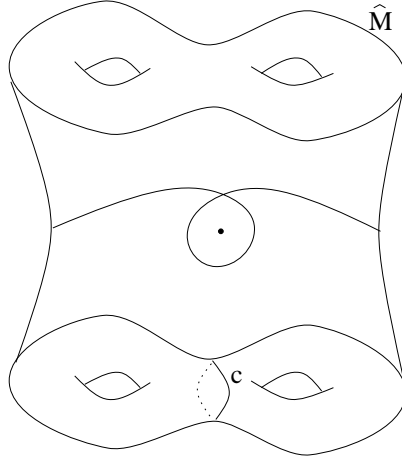


Figure 1: The surface  $S$  is immersed in  $\hat{M}$  and is not homotopic to an embedding.

erations will show that the image of  $f_\rho$  misses  $c_\rho$  so we can view  $f_\rho$  as a map to  $\hat{M}$ . In particular, we can compare the homotopy classes of the maps  $f_\rho$  in  $\hat{M}$ .

The heart of the proof is that we can find representations  $\rho_0$  and  $\rho_1$  in  $V \cap QF(S)$ , such that  $f_0$  is homotopic in  $\hat{M}$  to the original immersion  $f$  while  $f_1$  is homotopic to an embedding. If  $\rho_0$  and  $\rho_1$  are in the same component of  $V \cap QF(S)$  then our smoothly varying family of hyperbolic structures  $N_\rho$  will define a homotopy between  $f_0$  and  $f_1$  in  $\hat{M}$ . This contradiction proves the theorem.

To find the representation  $\rho_0$  we take a small deformation of  $\hat{M}$  that fills in the torus boundary to give a manifold homeomorphic to  $M$ . To find  $\rho_1$  we take a small deformation of  $M_\infty$  that resolves the rank one cusp. In this case the homeomorphism type is preserved. Since  $M_0$  is geometrically very close to  $\hat{M}$ ,  $f_0$  will have the same homotopy class as  $f$ . In  $M_\infty$ ,  $f$  lifts to an embedding and therefore  $f_1$  will be an embedding in  $M_1$ .

It is worthwhile to compare this result with the bumping of distinct components examined in [1] and [2]. As mentioned above in [2], necessary and sufficient conditions are given for components to bump. We will not state them here, but at the very least we need a manifold with more topology than an  $I$ -bundle so that the interior of  $AH(\pi_1(N))$  will have more than one component. The construction of the bumping representation is then very similar to the one above.

We first remove a suitably chosen simple closed curve  $c$  from  $M = \text{int } N$  to obtain a new manifold  $\hat{M}$ . We then find a cover  $M_\infty$  of  $\hat{M}$  that is homotopy equivalent, but in this case not homeomorphic to,  $M$ . A hyperbolic structure on  $\hat{M}$  defines a hyperbolic structure on  $M_\infty$ . As above we make a small deformation  $M_0$  of  $\hat{M}$  that will be homeomorphic to  $M$  while a small deformation  $M_1$  of  $M_\infty$  will be homeomorphic to  $M_\infty$ . Although  $M_0$  and  $M_1$  are not homeomorphic, their holonomy representations  $\rho_0$  and  $\rho_1$  will both be near the holonomy representation  $\rho_\infty$  of  $M_\infty$ . The next, and last, step is the real difference between the two arguments. As the components of the interior of  $AH(\pi_1(N))$  are parameterized by the (marked) oriented homeomorphism types of  $N$ ,  $\rho_0$  and  $\rho_1$  must be in distinct components that bump at  $\rho_\infty$ .

### Acknowledgments.

The authors would like to thank Jeff Brock and Dick Canary for interesting and helpful discussions.

## 2 Preliminaries

A *Kleinian group* is a discrete, torsion free subgroup of the orientation preserving isometries of hyperbolic 3-space,  $\mathbb{H}^3$ . In the upper-half-space model of  $\mathbb{H}^3$  the orientation-preserving isometries are identified with the group  $PSL_2(\mathbb{C})$ , so that a Kleinian group can be considered a discrete, torsion free subgroup of  $PSL_2(\mathbb{C})$ .

Let  $\Gamma$  be a Kleinian group and set  $M$  to be the quotient manifold  $\mathbb{H}^3/\Gamma$ . The *convex core* of  $M$  is the smallest convex submanifold of  $M$  whose inclusion in  $M$  is a homotopy equivalence. If the convex core has finite volume, and  $\Gamma$  is finitely generated then  $\Gamma$  is called *geometrically finite*. In addition, a geometrically finite Kleinian group is *minimally parabolic* if every maximal parabolic subgroup is of rank 2.

Let  $R(\pi_1(N)) = \text{Hom}(\pi_1(N), PSL_2(\mathbb{C}))/PSL_2(\mathbb{C})$  be the space of conjugacy classes of representations of  $\pi_1(N)$  in  $PSL_2(\mathbb{C})$  where  $N$  is a compact, orientable, atoroidal 3-manifold. The subset  $AH(\pi_1(N)) \subset R(\pi_1(N))$  consists of the discrete, faithful representations of  $\pi_1(N)$ , modulo conjugacy. It is a result of Jørgensen [9] that  $AH(\pi_1(N))$  is a closed subset of  $R(\pi_1(N))$ . By work of Marden [10] and Sullivan [12] the interior of  $AH(\pi_1(N))$  is  $MP(\pi_1(N))$ , the minimally parabolic representations.

A representation  $\rho \in AH(\pi_1(N))$  determines an oriented hyperbolic manifold  $M_\rho = \mathbb{H}^3/\rho(\pi_1(N))$  along with a homotopy equivalence,  $f_\rho : N \longrightarrow M_\rho$ . While in general  $MP(\pi_1(N))$  will have many components, in this paper

our interest is the topology of the closure of a single component  $B$ . Note that  $AH(\pi_1(N))$  is determined only by the homotopy type of  $N$ . We can therefore assume that  $N$  is chosen such that if  $\rho$  is in  $B$  then there is a homeomorphism from  $M_\rho$  to the interior of  $N$  that is a homotopy inverse for  $f_\rho$ . We can also orient  $N$  such that this homeomorphism is orientation preserving. Then  $B$  will be the unique component of  $MP(\pi_1(N))$  satisfying these two properties.

We also need to work with hyperbolic structures on the compact manifold  $N$  that may not extend to complete hyperbolic structures on an open manifold containing  $N$ . We let  $\mathcal{H}(N)$  be the space of hyperbolic metrics on  $N$ . Given two hyperbolic metrics on  $N$  the identity map will be a biLipschitz map between the two metrics. Given a structure,  $N' \in \mathcal{H}(N)$ , a neighborhood  $N'(\epsilon)$  of  $N'$  consists of those structures in  $\mathcal{H}(N)$  for which the identity map from  $N'$  is a  $(1+\epsilon)$ -biLipschitz map. The  $N'(\epsilon)$  are a basis of neighborhoods for  $N'$ .

Theorem 1.7.1 in [6] describes the local structure of a neighborhood of  $N'$ . We will need the following simple consequence of this theorem:

**Theorem 2.1** [6] *The holonomy map  $\mathcal{H}(N) \longrightarrow R(\pi_1(N))$  is locally onto. Furthermore, for any neighborhood  $V$  of  $N'$  there exists a neighborhood  $U \subset V$ , such that if  $N_0$  and  $N_1$  are hyperbolic structures in  $U$  with holonomy  $\rho_0$  and  $\rho_1$ , respectively, and  $\rho_t$ ,  $0 \leq t \leq 1$ , is a path in the image of  $U$  then there is a path  $N_t$  in  $U$ , where each  $N_t$  has holonomy  $\rho_t$ .*

Now assume that  $\partial N$  contains at least one torus,  $T$ . Choose a meridian and longitude for this torus such that elements of  $\pi_1(T) = \mathbb{Z} \oplus \mathbb{Z}$  are determined by a pair of integers. Let  $(p, q)$  be a pair of relatively prime integers. Let  $N(p, q)$  denote the result of performing  $(p, q)$ -Dehn filling on  $N$  along this torus; that is, there exists an embedding  $d_{p,q} : N \longrightarrow N(p, q)$  such that  $\overline{N(p, q) - d_{p,q}(N)}$  is a solid torus bounded by  $d_{p,q}(T)$  and the image of the  $(p, q)$  curve on  $T$  is trivial in  $N(p, q)$ . Let  $\gamma$  denote the core curve of the solid torus. If  $N$  and  $N(p, q)$  have complete hyperbolic structures,  $M$  and  $M(p, q)$ , on their interiors then  $M(p, q)$  is a *hyperbolic Dehn filling* of  $M$  if  $M(p, q) - d_{p,q}(M)$  contains the geodesic representative of  $\gamma$ . Note that a hyperbolic structure  $M(p, q)$  may not be a hyperbolic Dehn filling of  $M$  if  $\gamma$  is not isotopic to its geodesic representative. Also note that the holonomy representation  $\rho$  for  $M(p, q)$  induces a non-faithful, holonomy representation,  $\rho_{p,q}$ , for  $N$  via pre-composition with  $(d_{p,q})_*$ .

If  $N$  has  $k$  torus boundary components, we can Dehn fill each of them. Let relatively prime integers,  $(p_i, q_i)$ , be the Dehn filling coefficients for the  $i$ -

th torus and let  $(\mathbf{p}, \mathbf{q}) = (p_1, q_1; \dots; p_k, q_k)$ . Then  $N(\mathbf{p}, \mathbf{q})$  is the  $(\mathbf{p}, \mathbf{q})$ -Dehn filling of  $N$ .

The following theorem has an extensive history. The interested reader should also see [3], [13], [4], and [7].

**Theorem 2.2** *The Hyperbolic Dehn Surgery Theorem ([5])*

*Let  $M$  be a compact 3-manifold with  $k$  torus boundary components and assume  $M$  has a minimally parabolic hyperbolic structure with holonomy  $\rho$ . We then have the following:*

1. *Except for a finite number of pairs for each  $i = 1, \dots, k$ , for each collection of relatively prime pairs  $(\mathbf{p}, \mathbf{q})$  there exist a geometrically finite hyperbolic  $(\mathbf{p}, \mathbf{q})$ -Dehn filling  $M(\mathbf{p}, \mathbf{q})$  of  $M$ .*
2.  $\rho_{\mathbf{p}, \mathbf{q}} \rightarrow \rho$  as  $|\mathbf{p}, \mathbf{q}| \rightarrow \infty$  ( $|\mathbf{p}, \mathbf{q}| = |p_1| + |q_1| + \dots + |p_k| + |q_k|$ ).
3. *If  $X$  is the complement of a neighborhood of the cusps and  $|\mathbf{p}, \mathbf{q}| > n$  then  $d_{\mathbf{p}, \mathbf{q}}|_X$  is  $K_n$ -biLipschitz with  $K_n \rightarrow 1$  as  $n \rightarrow \infty$ .*

### 3 Wraps and twists

Let

$$X = [-1, 1] \times [-1, 1] \times S^1$$

and

$$\hat{X} = X - \left( \left[-\frac{1}{3}, \frac{1}{3}\right] \times \left[-\frac{1}{3}, \frac{1}{3}\right] \times S^1 \right).$$

We begin by defining maps of the annulus,

$$A = [-1, 1] \times S^1$$

into  $\hat{X} \subset X$ . First we define  $w : A \rightarrow \hat{X}$  by

$$w(x, \theta) = \left( -\frac{1}{2} \sin(\pi x), \frac{1}{2} \cos(\pi x), \theta \right).$$

We next define a sequence of maps  $w_n : A \rightarrow \hat{X}$  for each  $n > 0$ . For each  $t$  and  $t'$  with  $-1 \leq t < t' \leq 1$  we let  $h_{t, t'} : ([t, t'] \times S^1) \rightarrow A$  be a homeomorphism that satisfies the conditions,  $h_{t, t'}(t, \theta) = (-1, \theta)$  and  $h_{t, t'}(t', \theta) = (1, \theta)$ . To define  $w_n$  we choose real numbers,  $t_0, \dots, t_n$  with  $-\frac{1}{3} = t_0 < t_1 < \dots < t_n = \frac{1}{3}$ , and let

$$w_n(x, \theta) = \begin{cases} \left( \frac{3}{2}x + \frac{1}{2}, -\frac{1}{2}, \theta \right) & \text{if } -1 \leq x < -\frac{1}{3} \\ w \circ h_{t_i, t_{i+1}} & \text{if } t_i \leq x < t_{i+1} \\ \left( \frac{3}{2}x - \frac{1}{2}, -\frac{1}{2}, \theta \right) & \text{if } \frac{1}{3} \leq x \leq 1. \end{cases}$$

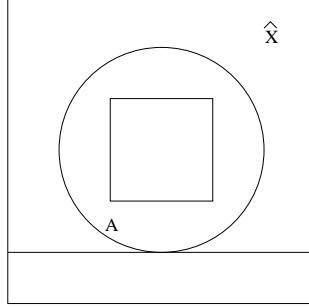


Figure 2: The image of  $A$  under the map  $w_1$  in a cross section of  $\hat{X}$ .

The map  $w_n$  wraps the annulus  $n$  times around the missing core of  $\hat{X}$ . For  $n = 0$ , we define  $w_0$  by  $w_0(x, \theta) = (x, -1/2, \theta)$ .

Our next family of maps,  $t_{n,m} : \hat{X} \rightarrow \hat{X}$ , are homeomorphisms which *Dehn twist*  $\hat{X}$ . They are defined by the following formula:

$$t_{n,m} = \begin{cases} (x, y, \theta) & \text{if } -1 \leq x < -\frac{1}{3} \text{ or } \frac{1}{3} < x \leq 1 \\ (x, y, \theta + 3n\pi(x + \frac{1}{3})) & \text{if } -\frac{1}{3} \leq x \leq \frac{1}{3} \text{ and } y > \frac{1}{3} \\ (x, y, \theta + 3m\pi(x + \frac{1}{3})) & \text{if } -\frac{1}{3} \leq x \leq \frac{1}{3} \text{ and } y < -\frac{1}{3}. \end{cases}$$

**Lemma 3.1** *The maps  $w_n$  and  $t_{k(n+1),kn} \circ w_n$  are homotopic rel  $\partial A$  for any positive integer  $n$  and any integer  $k$ .*

**Proof**

Let  $\hat{X}_{\frac{1}{3}} = ([-\frac{1}{3}, \frac{1}{3}] \times [-1, 1] \times S^1) \cap \hat{X}$  denote the middle-third of  $\hat{X}$ ; it has two components, the upper half and the lower half. The image of  $A$  under the map  $w_n$  intersects  $\hat{X}_{\frac{1}{3}}$ , so that  $w_n^{-1}(w_n(A) \cap \hat{X}_{\frac{1}{3}})$  consists of  $2n + 1$  essential sub-annuli of  $A$ ;  $n$  of the annuli map into the upper half of the middle third, while  $n + 1$  of the annuli map into the lower half. On the each of the  $n + 1$  annuli mapping into the lower half,  $t_{k(n+1),kn}$  is a  $kn$ -Dehn twist, while on the  $n$  upper annuli  $t_{k(n+1),kn}$  is a  $-k(n + 1)$ -Dehn twist. Therefore the total affect of  $t_{k(n+1),kn}$  is a  $kn(n + 1) - k(n + 1)n = 0$ -Dehn twist and  $w_n$  is homotopic to  $w_n \circ t_{k(n+1),kn}$  rel  $\partial A$  (see Figure 3).

proof of Lemma 3.1

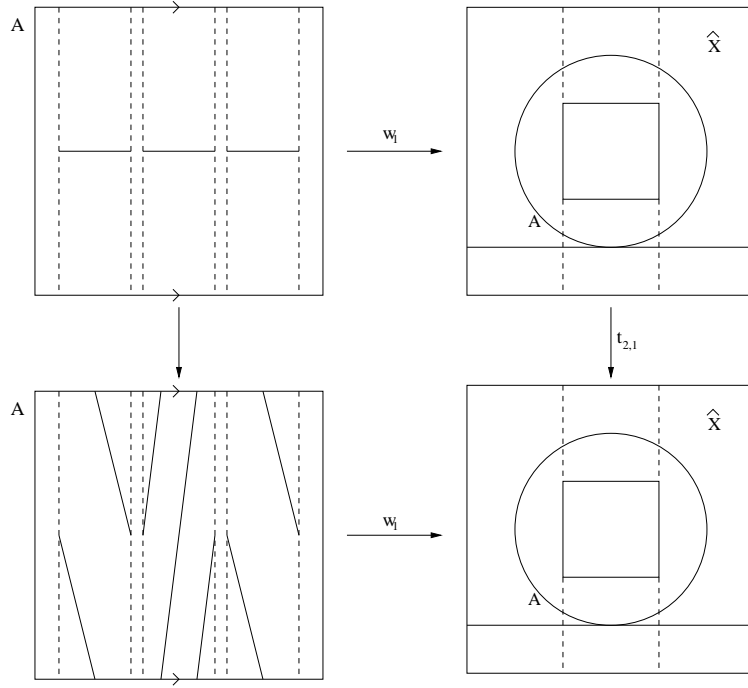


Figure 3: By identifying the top and bottom of the squares on the left we obtain (two copies of) the annulus  $A$ . The preimage of the  $w_1(A) \cap \hat{X}_{\frac{1}{3}}$  is the three dashed annuli. The effect of  $t_{2,1}$  on  $A$ , is two dehn twists on the center annuli and a single dehn twist in the opposite direction on the two outside annuli. As we see from the picture in the lower left, the net effect on  $A$  is a map that is homotopic to the identity.

We now relate the maps  $t_{n,m}$  to the Dehn filling of  $\hat{X}$ . As our coordinates for Dehn filling we choose the meridian to be the unique homotopy class that is trivial in  $X$  and the longitude to be the curve  $\{\frac{1}{3}\} \times \{\frac{1}{3}\} \times S^1$ . Recall the Dehn filling maps  $d_{1,k} : \hat{X} \rightarrow \hat{X}(1,k)$ .

**Lemma 3.2** *For each  $t_{n,m}$  there exists a homeomorphism  $h_{n,m} : \hat{X}(1,n-m) \rightarrow \hat{X}(1,0)$  such that  $d_{1,0} \circ t_{n,m} = h_{n,m} \circ d_{1,n-m}$ .*

**Proof**

The map  $t_{n,m}$  takes the  $(1,n-m)$ -curve to the  $(1,0)$ -curve so  $d_{1,0} \circ t_{n,m}$  takes the  $(1,n-m)$  to a trivial curve in  $\hat{X}(1,0)$ . On the image of  $\hat{X}$  in  $\hat{X}$ , we define  $h_{n,m}$  to satisfy the equation,  $d_{1,0} \circ t_{n,m} = h_{n,m} \circ d_{1,n-m}$ . Since the



image of the  $(1, n - m)$  curve is trivial in  $\hat{X}(1, n - m)$ ,  $h_{n,m}$  extends to a homeomorphism.

proof of Lemma 3.2

Let

$$\partial_0 X = [-1, 1] \times \{-1, 1\} \times S^1 \subset X$$

and

$$\partial_1 X = \{1\} \times [-1, 1] \times S^1 \subset X.$$

Also assume that  $N$  is a compact manifold with boundary and that  $N$  contains an essential, boundary incompressible annulus. Then there is a pairwise embedding of  $(X, \partial_0 X)$  in  $(N, \partial N)$  such that  $\partial_1 X$  is an essential, boundary incompressible annulus. Identify  $A$  with the lower half of  $\partial_0 X$ ; that is, the annulus  $[-1, 1] \times \{-1\} \times S^1$ . Let  $c = \{0\} \times \{0\} \times S^1$  be the core curve of  $X$  and let  $\hat{M} = M - c$  where  $M$  is the interior of  $N$ .

For each integer  $n \geq 0$  we define an immersion  $s_n : N \rightarrow M \subset N$  as follows. The map  $s_n$  is homotopic to the identity map and a homeomorphism onto its image outside of  $X$ . We also require that  $s_n(N) \cap c = \emptyset$  and that  $s_n$  restricted to  $A$  is homotopic to  $w_n$  rel  $\partial A$ . This completely defines  $s_n$  up to homotopy in  $\hat{M}$ . We call any map that satisfies these properties a *shuffle immersion*.

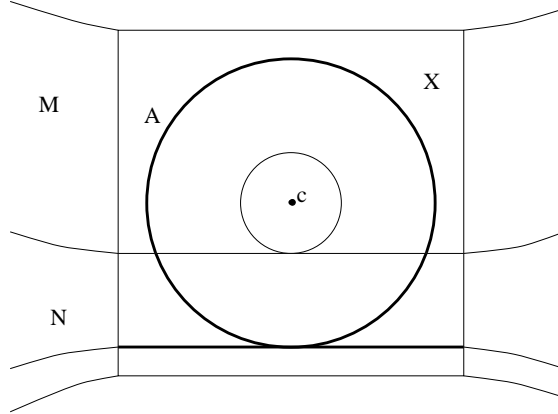


Figure 4: The map  $s_1$  immerses  $N$  in  $M$  and is not homotopic to an embedding in  $\hat{M}$ .

**Lemma 3.3** *A shuffle immersion  $s_n$  satisfies the following properties:*

1. If  $n \neq m$  then  $s_n$  and  $s_m$  are not homotopic in  $\hat{M}$ .
2. For each integer  $k$ , there is an orientation preserving homeomorphism  $h_k : \hat{M}(1, k) \longrightarrow M$  such that  $s_n$  and  $h_k \circ d_{1,k} \circ s_n$  are homotopic in  $\hat{M}$ . Here,  $M = \hat{M}(1, 0)$ .
3. The cover of  $\hat{M}$  associated to  $(s_n)_*(\pi_1(N))$  is homeomorphic to  $M$  and  $s_n$  lifts to an embedding  $\hat{s}_n : N \longrightarrow M$  which is homotopic to  $s_0$  in  $\hat{M}$ .

**Proof**

1. If  $n \neq m$ , the maps  $(s_n)_*(\pi_1(N))$  and  $(s_m)_*(\pi_1(N))$  are non-conjugate subgroups of  $\pi_1(\hat{M})$  and therefore the maps  $s_n$  and  $s_m$  are not homotopic.
2. On  $\hat{X}(1, k) \subset \hat{M}(1, k)$  we let  $h_k = h_{k(n+1), kn}$ . Using Lemma 3.2, we see that  $h_k$  extends to a homeomorphism from  $\hat{M}(1, k)$  to  $M$ . By Lemma 3.1,  $s_n$  and  $h_k \circ d_{1,k} \circ s_n$  are homotopic in  $\hat{M}$ .
3. This is an easy exercise in 3-manifold topology which we leave to the reader.

proof of Lemma 3.3

## 4 Self-bumping

We now use the topology we developed in §3. With the same assumptions as in §3 we fix a shuffle immersion  $f = s_d$ , with  $d > 0$ . Note that such a shuffle immersion exists if and only if  $N$  contains an essential, boundary incompressible annulus. However, for  $M$  and  $\hat{M}$  to support complete hyperbolic structures we need to make further topological restrictions. Namely,  $N$  must be irreducible and atoroidal and the simple closed curve  $c$  must be primitive and not homotopic to a torus boundary component of  $\partial N$ . Then  $M$  and  $\hat{M}$  satisfies the conditions of Thurston's hyperbolization theorem (see Lemma 2.5.10 in [8]) and we fix a minimally parabolic hyperbolic structure  $\hat{M}_\infty$  on  $\hat{M}$  with holonomy representation  $\hat{\rho}_\infty$ . We also let  $N_\infty$  be the hyperbolic metric on  $N$  obtained as the pull-back by  $f$  of the metric  $\hat{M}_\infty$  on  $\hat{M}$ .

We now set up a notational system that will hold for the remainder of the paper. For an index  $\alpha$ ,  $N_\alpha$  is a hyperbolic structure on  $N$  and  $\rho_\alpha$  will be the associated holonomy representation. Let  $M$  be the interior of

$N$ . As we noted in the introduction, if  $\rho_\alpha \in AH(\pi_1(N))$  then  $M_\alpha$  is a complete hyperbolic structure, marked by  $N$ . As  $N_\alpha$  has the same holonomy as  $M_\alpha$  there will be an isometric immersion,  $f_\alpha : N_\alpha \longrightarrow M_\alpha$ , with  $f_\alpha$  a homotopy equivalence. In other words,  $f_\alpha$  is the marking map. Let  $c_\alpha$  denote the geodesic representative of  $c$  in  $M_\alpha$ .

**Lemma 4.1** *Let  $V$  be a small neighborhood of  $\rho_\infty$ . Then for each  $N_\alpha$  near  $N_\infty$  with  $\rho_\alpha \in V \cap MP(\pi_1(N))$ ,  $f_\alpha(N_\alpha) \cap c_\alpha = \emptyset$ .*

**Proof**

By compactness there exists a  $K$  such that for any  $p \in N$  we can find a non-trivial simple closed curve  $\gamma_p$  through  $p$ , and not homotopic to  $c$ , with length  $< K$  in  $N_\infty$ . We choose  $V$  small enough such that all structures in the neighborhood are 2-biLipschitz from  $N_\infty$ . The Margulis lemma implies that there exists an  $\epsilon$  such that, for any complete hyperbolic 3-manifold, if a homotopically non-trivial simple closed curve intersects a homotopically distinct geodesic of length  $< \epsilon$  it has length  $> 3K$ . Furthermore, since the length of curves is continuous on  $R(\pi_1(N))$ , we can further shrink  $V$  so that the curve,  $c_\alpha$ , has length  $< \epsilon$  and therefore  $f_\alpha(\gamma_p)$ , which has length  $< 2K$ , does not intersect  $c_\alpha$ ; implying that  $p \notin c_\alpha$ .

proof of Lemma 4.1

Recall that  $B$  is a component of  $MP(\pi_1(N))$  so that the marking map  $f_\alpha$  has a homotopy inverse which is an orientation preserving homeomorphism between  $M_\alpha$  and  $M$  if and only if  $\rho_\alpha \in B$ .

**Lemma 4.2** *For the shuffle immersion  $f$ , there exists a sequence of hyperbolic structures  $N_k$  with holonomy representations  $\rho_k$ , such that:*

1.  $N_k \rightarrow N_\infty$ .
2.  $\rho_k \rightarrow \rho_\infty$ .
3. *There exist homeomorphisms  $h_k : M_k \longrightarrow M$  such that  $h_k$  is a homotopy inverse for  $f_k|_M$ ,  $h_k(c_k) = c$  and  $f$  and  $h_k \circ f_k$  are homotopic in  $\hat{M}$ . In particular,  $\rho_k \in B$ .*

**Proof**

1. For large  $n$ , let  $M_n = \hat{M}_\infty(1, n)$  be the manifolds obtained by performing hyperbolic Dehn surgery on  $\hat{M}_\infty$  as in Theorem 2.2. Since

$f_\infty(N_\infty)$  is contained in a compact subset of  $\hat{M}$ , Theorem 2.2 also implies that the maps  $d_{1,n} : \hat{M}_\infty \rightarrow M_n$  restricted to  $f_\infty(N_\infty)$  are  $K_n$ -quasi-isometries with  $K_n \rightarrow 1$  as  $n \rightarrow \infty$ . The hyperbolic structures  $N_n$  defined by pulling back the hyperbolic metric on  $M_n$  by  $d_{1,n} \circ f_\infty$  converge to  $N_\infty$ .

2. Again, as in Theorem 2.2, the maps  $d_{1,n}$  define holonomy representations  $\hat{\rho}_{1,n}$  of  $\pi_1(\hat{M})$  with  $\hat{\rho}_{1,n} \rightarrow \hat{\rho}_\infty$ . The holonomy representations,  $\rho_n$ , are restrictions of  $\hat{\rho}_{1,n}$  so  $\rho_n \rightarrow \rho_\infty$ .

3. These homeomorphisms are supplied by Lemma 3.3.

proof of Lemma 4.2

The following lemma will be used to detect when two representations are not contained in the same component of  $V \cap B$ .

**Lemma 4.3** *Let  $U$  be a neighborhood of  $N_\infty$  that satisfies the conclusion of Theorem 2.1 and Lemma 4.1 and let  $V$  be the image of  $U$  under the holonomy map. Let  $N_0$  and  $N_1$  be hyperbolic structures in  $U$  with holonomy  $\rho_0$  and  $\rho_1$ , both in  $V \cap MP(\pi_1(N))$ . Also assume that  $h_i : M_i \rightarrow M$ ,  $i = 0, 1$ , are homeomorphisms that are homotopy inverses of  $f_i|_M$  and  $h_i(c_i) = c$ . If  $\rho_t$ ,  $0 \leq t \leq 1$ , is a path in  $V \cap MP(\pi_1(M))$  then  $h_0 \circ f_0$  and  $h_1 \circ f_1$  are homotopic in  $\hat{M}$ .*

**Proof**

By Theorem 2.1 we have a path of structures  $N_t$  in  $V$  with holonomy  $\rho_t$ . The  $\rho_t$  are in the same component of  $MP(\pi_1(M))$  as  $\rho_0$  and  $\rho_1$  are in, so there are homeomorphisms,  $h_t : M_t \rightarrow M$ , that are homotopy inverses of  $f_t|_M$ . We can assume that the push-forward of the hyperbolic metrics on  $M_t$  to  $M$  is a continuously changing family of metrics on  $M$ . Furthermore, as all the  $c_t$  are short geodesics, they will be simple. Hence  $h_t(c_t)$  is an isotopy of  $c$  in  $M$ . We can therefore modify the  $h_t$  such that  $h_t(c_t) = c$ . Then  $h_t \circ f_t$  will vary continuously in  $t$ . By Lemma 4.1  $f_t(N_t) \cap c_t = \emptyset$  so  $h_t \circ f_t$  is a homotopy between  $h_0 \circ f_0$  and  $h_1 \circ f_1$  in  $\hat{M}$ .

proof of Lemma 4.3

We next apply Lemma 4.3 to show that distinct shuffle immersion force  $V \cap B$  to be disconnected.

**Lemma 4.4** *Let  $f, f' : N \longrightarrow \hat{M} \subset M$ , be distinct shuffle immersions. Assume that there exists minimally parabolic structures  $\hat{M}_\infty$  and  $\hat{M}'_\infty$  on  $\hat{M}$  such that the pulled-back hyperbolic structures  $N_\infty$  and  $N'_\infty$  are isometric and hence define the same holonomy representation,  $\rho_\infty$ . Then for every small neighborhood  $V$  of  $\rho_\infty$ ,  $V \cap B$  is disconnected.*

**Proof**

Let  $M_n, N_n, f_n, h_n$ , and  $\rho_n$  and  $M'_n, N'_n, f'_n, h'_n$ , and  $\rho'_n$  be the hyperbolic structures, isometric immersions and holonomy representations given by Lemma 4.2 for  $f$  and  $f'$ , respectively. Choose an open neighborhood  $V$  of  $\rho_\infty$  given by Lemma 4.1.

There exists integers  $n$  and  $m$  such that  $\rho_n, \rho'_m \in V$ . The intersection  $V \cap B$  is an open subset of the manifold  $B$  so the connected components of  $V \cap B$  are path connected. If  $\rho_n$  and  $\rho'_m$  are in the same component of  $V \cap B$  then Lemma 4.3 implies that  $h_n \circ f_n$  and  $h'_m \circ f'_m$  are homotopic in  $\hat{M}$ . On the other hand, by Lemma 4.2,  $h_n \circ f_n$  and  $h'_m \circ f'_m$  are homotopic in  $\hat{M}$  to  $f$  and  $f'$ , respectively. Since,  $f$  and  $f'$  aren't homotopic in  $\hat{M}$  we have a contradiction.

proof of Lemma 4.4

We now prove our main theorem.

**Theorem 4.5** *Let  $N$  be a compact, orientable, atoroidal, irreducible 3-manifold with boundary. Suppose that  $N$  contains an essential, boundary incompressible annulus whose core curve is not homotopic into a torus boundary component of  $\partial N$ . Let  $B$  be a component of the interior of  $AH(\pi_1(N))$ . Then there is a representation  $\rho$  in  $\overline{B}$  such that for any sufficiently small neighborhood  $V$  of  $\rho$  in  $AH(\pi_1(N))$  the set  $V \cap B$  is disconnected.*

**Proof**

We recall our standing assumption that if  $\rho \in B$  then the marking map  $f_\rho : N \longrightarrow M_\rho$  has a homotopy inverse that is a homeomorphism onto the interior of  $N$ . If we want to show self-bumping at a different component  $B'$  we find a new manifold  $N'$  homotopy equivalent to  $N$  such that  $N'$  and  $B'$  have the above property. With the exceptions of  $N$  being irreducible and atoroidal, all of the topological assumptions we have made depend only on the homotopy type of  $N$ . Since a hyperbolic manifold is automatically irreducible and atoroidal,  $N'$  will also be atoroidal, irreducible and contain

an essential, boundary incompressible annulus. In particular if one component of  $MP(\pi_1(N))$  self-bumps then every component of  $MP(\pi_1(N))$  will self-bump.

By Lemma 3.3, there is a non-trivial shuffle immersion  $f : N \longrightarrow \hat{M} \subset M$  and  $f$  lifts to an embedding  $f'$  in the cover  $M'$  associated to  $f_*(\pi_1(N))$ , with  $M'$  homeomorphic to  $M$ . Let  $\hat{M}_\infty$  be a minimally parabolic structure on  $\hat{M}$  which defines a hyperbolic structure  $M'_\infty$  on  $M' = M$ . We use  $f$  to pull back a hyperbolic structure  $N_\infty$  on  $N$  and then  $f_\infty : N_\infty \longrightarrow \hat{M}_\infty$  is an isometric immersion and  $f'_\infty : N_\infty \longrightarrow M'_\infty$  is an isometric embedding. The holonomy,  $\rho_\infty(c)$ , of  $c$  will be parabolic so by an application of the second Klein-Maskit combination we can find another parabolic  $\gamma$  such that the free product of  $\rho_\infty(\pi_1(N))$  and  $\gamma$  is a uniformization  $\hat{M}'_\infty$  of  $\hat{M}$  such that  $M'_\infty$  covers  $\hat{M}'_\infty$  and  $f'_\infty$  descends to an embedding. Therefore  $f$  and  $f'$  satisfy the conditions of Lemma 4.4 which implies the theorem.

proof of Theorem 4.5

**Corollary 4.6**  $\overline{B}$  is not a manifold.

**Proof**

If  $\overline{B}$  is a manifold then Theorem 4.5 implies that  $\rho_\infty$  is in the interior of  $\overline{B}$ , since it cannot be in the boundary. However, in [12], Sullivan proves that the interior of  $\overline{B}$  is  $B$ . Since  $\rho_\infty$  is not in  $B$ ,  $\overline{B}$  is not a manifold.

proof of Corollary 4.6

In Theorem 4.5 we characterized when the components of  $MP(\pi_1(N))$  self-bump. To do so we constructed a representation where this self-bumping occurs. In our next theorem we describe a sufficient condition for a representation to be a point of self-bumping. To describe it we will assume some knowledge of Kleinian groups.

We now allow  $N$  to contain more than one copy of  $X$ . In particular, assume that there are  $m$  disjoint, pairwise embeddings of  $(X, \partial X)$  in  $(N, \partial N)$ , labeled,  $X_1, \dots, X_m$ . As before we assume that each  $\partial_1 X_i$  is an essential, boundary incompressible annulus and that each core curve,  $c_i$  is primitive and not homotopic to a boundary torus. We further assume that the  $c_i$  are homotopically distinct. For each  $i$ ,  $1 \leq i \leq m$ , choose an integer,  $n_i \geq 0$ . There is then a shuffle immersion,  $s_{n_1, \dots, n_m}$ , that wraps  $N$  around  $c_i$ ,  $n_i$  times.

Let  $\hat{M} = M - \mathcal{C}$ . If  $\hat{\rho}$  is a minimally parabolic, geometrically finite uniformization of  $\hat{M}$  then the space of all minimally parabolic hyperbolic

structures on  $\hat{M}$ , with the same marking, is  $QD(\hat{\rho})$ , the *quasiconformal deformation space* of  $\hat{\rho}$ . The image of  $(s_{n_1, \dots, n_m})_*(\pi_1(N))$  in  $\pi_1(\hat{M})$  defines a Kleinian subgroup  $\Gamma$  of  $\hat{\Gamma} = \hat{\rho}(\pi_1(\hat{M}))$  that uniformizes  $M$ , and a representation  $\rho = \hat{\rho} \circ (s_{n_1, \dots, n_m})_*$ , with image  $\Gamma$ . If  $\hat{\rho}'$  is another representation in  $QD(\hat{\rho})$  then  $\hat{\rho}' \circ (s_{n_1, \dots, n_m})_*$  is in  $QD(\rho)$ , the quasiconformal deformation space of  $\rho$ . Therefore  $(s_{n_1, \dots, n_m})_*$  defines a map between  $QD(\hat{\rho})$  and  $QD(\rho)$ . Our previous work shows the following:

**Theorem 4.7** *All representations in  $QD(\rho)$  in the image of  $QD(\hat{\rho})$  under  $(s_{n_1, \dots, n_m})_*$  are points of self-bumping for  $B$  if  $n_i \neq 0$  for some  $i$ .*

Note that  $\rho$  will not be minimally parabolic, for the  $c_i$  will all be parabolic in  $\Gamma = \rho(\pi_1(N))$ . Let  $c'_i = \{0\} \times \{1\} \times S^1 \subset \partial_0 X_i$ . The quotient of the domain of discontinuity for  $\Gamma$  will be a conformal structure on  $\partial N - \coprod c'_i$ . As the pinched curves in  $\partial N$  are determined by the embeddings of the  $X_i$ , if  $s_{n'_1, \dots, n'_m}$  is another shuffle immersion then the image of  $(s_{n'_1, \dots, n'_m})_*$  will be the same quasiconformal deformation space,  $QD(\rho)$ . (While these maps have the same range,  $(s_{n_1, \dots, n_m})_*(\hat{\Gamma}) \neq (s_{n'_1, \dots, n'_m})_*(\hat{\Gamma})$ .) On the other hand, each  $X_i$  has an involution which swaps the two components of  $\partial_0 X_i$ . By performing this involution on some (possibly all) of the  $X_i$  we get a new family of shuffle immersions. The bumping representations associated to these shuffle immersions will then lie in a different quasi-conformal deformation space.

We also remark that even in the case where  $N$  is an  $I$ -bundle, Theorem 4.7 is stronger than McMullen's result in [11]. In McMullen's theorem, all the  $c'_i$  must lie in the same component of  $\partial N$ . Here we have no such restriction.

We close with the following conjecture.

**Conjecture 4.8** *A representation  $\rho$  is a point of self-bumping for  $B$  if and only if there is a non-empty collection of curves  $\mathcal{C}$  (as above) in  $M$ , a shuffle immersion  $s$  with respect to  $\mathcal{C}$ , and a uniformization  $\hat{\rho}$  of  $\hat{M} = M - \mathcal{C}$  so that  $\rho = \hat{\rho} \circ s_*$ .*

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